CS 188: Artificial Intelligence Spring 2011

Lecture 17: Bayes Nets V
3/30/2011

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Many slides over this course adapted from Dan Klein, Stuart Russell, Andrew Moore

Announcements

- Section
  - We'll be using some software to play with Bayes nets: Bring your laptop!
  - Download necessary files (links also in the handout): http://www-inst.eecs.berkeley.edu/~cs188/sp11/bayes/bayes.jar and http://www-inst.eecs.berkeley.edu/~cs188/sp11/bayes/network.xml
- Assignments
  - P4 and contest going out Monday

Outline

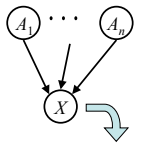
- Bayes net refresher:
  - Representation
  - Exact Inference
    - Enumeration
    - Variable elimination
- Approximate inference through sampling

Bayes' Net Semantics

- A set of nodes, one per variable X
- A directed, acyclic graph
- A conditional distribution for each node
  - A collection of distributions over X, one for each combination of parents' values

$$P(X|a_1 \dots a_n)$$

  - CPT: conditional probability table



$P(X|A_1 \dots A_n)$

A Bayes net = Topology (graph) + Local Conditional Probabilities

Probabilities in BNs

- For all joint distributions, we have (chain rule):

$$P(x_1, x_2, \dots x_n) = \prod_{i=1}^{n} P(x_i|x_1, \dots, x_{i-1})$$

- Bayes' nets implicitly encode joint distributions
  - As a product of local conditional distributions
  - To see what probability a BN gives to a full assignment, multiply all the relevant conditionals together:

$$P(x_1, x_2, \dots x_n) = \prod_{i=1}^{n} P(x_i|parents(X_i))$$

- This lets us reconstruct any entry of the full joint
- Not every BN can represent every joint distribution
  - The topology enforces certain conditional independencies

Inference by Enumeration

- Given unlimited time, inference in BNs is easy
- Recipe:
  - State the marginal probabilities you need
  - Figure out ALL the atomic probabilities you need
  - Calculate and combine them
- Building the full joint table takes time and space exponential in the number of variables

General Variable Elimination

- Query: $P(Q|E_1 = e_1, \ldots E_k = e_k)$
- Start with initial factors:
  - Local CPTs (but instantiated by evidence)
- While there are still hidden variables (not Q or evidence):
  - Pick a hidden variable H
  - Join all factors mentioning H
  - Eliminate (sum out) H
- Join all remaining factors and normalize
- Complexity is exponential in the number of variables appearing in the factors---can depend on ordering but even best ordering is often impractical
- Worst case is bad: we can encode 3-SAT with a Bayes net (NP-complete)

Approximate Inference

- Simulation has a name: sampling (e.g. predicting the weather, basketball games…)
- Basic idea:
  - Draw N samples from a sampling distribution S
  - Compute an approximate posterior probability
  - Show this converges to the true probability P
- Why sample?
  - Learning: get samples from a distribution you don't know
  - Inference: getting a sample is faster than computing the right answer (e.g. with variable elimination)

Sampling

- How do you sample?
  - Simplest way is to use a random number generator to get a continuous value uniformly distributed between 0 and 1 (e.g. random() in Python)
  - Assign each value in the domain of your random variable a sub-interval of [0,1] with a size equal to its probability
    - The sub-intervals cannot overlap

Sampling Example

- Each value in the domain of W has a sub-interval of [0,1] with a size equal to its probability

$P(W)$

W	P
sun	0.6
rain	0.1
fog	0.3
meteor	0.0

u is a uniform random value in $[0, 1]$

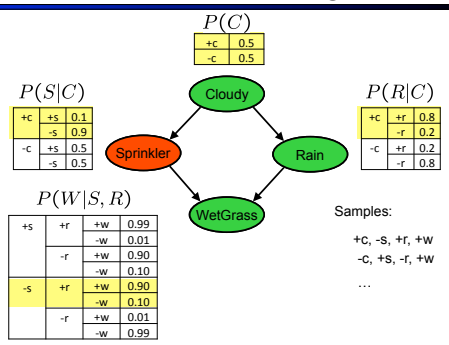
if $0.0 \leq u < 0.6$, $w =$ sun

if $0.6 \leq u < 0.7$, $w =$ rain

if $0.7 \leq u < 1.0$, $w =$ fog

e.g. if `random()` returns $u = 0.83$, then our sample is $w =$ fog

Prior Sampling



Prior Sampling

- This process generates samples with probability:

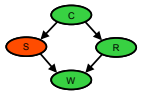
$$S_{PS}(x_1 \ldots x_n) = \prod_{i=1}^{n} P(x_i|\text{Parents}(X_i)) = P(x_1 \ldots x_n)$$

…i.e. the BN's joint probability

- Let the number of samples of an event be $N_{PS}(x_1 \ldots x_n)$
- Then $\lim_{N\to\infty} \hat{P}(x_1, \ldots, x_n) = \lim_{N\to\infty} N_{PS}(x_1, \ldots, x_n)/N$
  $= S_{PS}(x_1, \ldots, x_n)$
  $= P(x_1 \ldots x_n)$
- I.e., the sampling procedure is consistent

Example

- We'll get a bunch of samples from the BN:
  - +c, -s, +r, +w
  - +c, +s, +r, +w
  - -c, +s, +r, -w
  - +c, -s, +r, +w
  - -c, -s, -r, +w
- If we want to know P(W)
  - We have counts <+w:4, -w:1>
  - Normalize to get P(W) = <+w:0.8, -w:0.2>
  - This will get closer to the true distribution with more samples
  - Can estimate anything else, too
  - What about P(C| +w)? P(C| +r, +w)? P(C| -r, -w)?
  - Fast: can use fewer samples if less time (what's the drawback?)



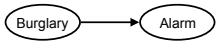
Rejection Sampling

- Let's say we want P(C)
  - No point keeping all samples around
  - Just tally counts of C as we go
- Let's say we want P(C| +s)
  - Same thing: tally C outcomes, but ignore (reject) samples which don't have S=+s
  - This is called rejection sampling
  - It is also consistent for conditional probabilities (i.e., correct in the limit)

+c, -s, +r, +w
+c, +s, +r, +w
-c, +s, +r, -w
+c, -s, +r, +w
-c, -s, -r, +w

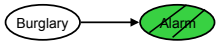
Likelihood Weighting

- Problem with rejection sampling:
  - If evidence is unlikely, you reject a lot of samples
  - You don't exploit your evidence as you sample
  - Consider P(B|+a)



-b, -a
-b, -a
-b, -a
-b, -a
+b, +a

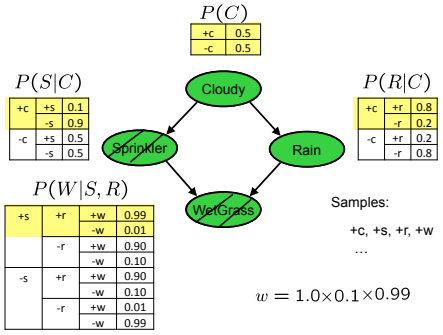
- Idea: fix evidence variables and sample the rest



-b +a
-b, +a
-b, +a
-b, +a
+b, +a

- Problem: sample distribution not consistent!
- Solution: weight by probability of evidence given parents

Likelihood Weighting



Samples:

+c, +s, +r, +w
…

$w = 1.0 \times 0.1 \times 0.99$

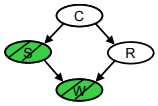
Likelihood Weighting

- Sampling distribution if z sampled and e fixed evidence

$$S_{WS}(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{l} P(z_i | \text{Parents}(Z_i))$$

- Now, samples have weights

$$w(\mathbf{z}, \mathbf{e}) = \prod_{i=1}^{m} P(e_i | \text{Parents}(E_i))$$



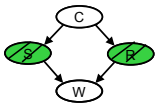
- Together, weighted sampling distribution is consistent

$$S_{\text{WS}}(z, e) \cdot w(z, e) = \prod_{i=1}^{l} P(z_i | \text{Parents}(z_i)) \prod_{i=1}^{m} P(e_i | \text{Parents}(e_i))$$

$$= P(\mathbf{z}, \mathbf{e})$$

Likelihood Weighting

- Likelihood weighting is good
  - We have taken evidence into account as we generate the sample
  - E.g. here, W's value will get picked based on the evidence values of S, R
  - More of our samples will reflect the state of the world suggested by the evidence
- Likelihood weighting doesn't solve all our problems
  - Evidence influences the choice of downstream variables, but not upstream ones (C isn't more likely to get a value matching the evidence)
- We would like to consider evidence when we sample every variable



Gibbs Sampling

- *Idea*: instead of sampling from scratch, create samples that are each like the last one.
- *Procedure*: resample one variable at a time, conditioned on all the rest, but keep evidence fixed.
- *Properties*: Now samples are not independent (in fact they're nearly identical), but sample averages are still consistent estimators!
- *What's the point*: both upstream and downstream variables condition on evidence.

Gibbs Sampling

- Say we want to sample P(S | R = +r)
- Step 1: Initialize
  - Set evidence (R = +r)
  - Set all other variables (S, C, W) to random values (e.g. by prior sampling or just uniformly sampling; say S = -s, W = +w, C = -c)
  - Our initial sample is then: (R = +r, S = -s, W = +w, C = -c)
- Steps 2+: Repeat the following for some number of iterations
  - Choose a non-evidence variable (S, W, or C in this case)
  - Sample this variable conditioned on nothing else changing
    - The first time through, if we pick S, we sample from P(S | R = +r, W = +w, C =-c)
  - The new sample can only be different in a single variable

Gibbs Sampling

- How is this better than sampling from the full joint?
  - In a Bayes net, sampling a variable given all the other variables (e.g. P(R|S,C,W)) is usually much easier than sampling from the full joint distribution
    - Only requires a join on the variable to be sampled (in this case, a join on R)
    - The resulting factor only depends on the variable's parents, its children, and its children's parents (this is often referred to as its Markov blanket)

Gibbs Sampling Example

- Want to sample from P(R | +s,-c,-w)
  - Shorthand for P(R | S=+s,C=-c,W=-w)

$$P(R|+s,-c,-w) = \frac{P(R,+s,-c,-w)}{P(+s,-c,-w)}$$
$$= \frac{P(R,+s,-c,-w)}{\sum_r P(R=r,+s,-c,-w)}$$
$$= \frac{P(-c)P(+s|-c)P(R|-c)P(-w|+s,R)}{\sum_r P(-c)P(+s|-c)P(R=r|-c)P(-w|+s,R=r)}$$
$$= \frac{P(R|-c)P(-w|+s,R)}{\sum_r P(R=r|-c)P(-w|+s,R=r)}$$

- Many things cancel out -- just a join on R!

Further Reading*

- Gibbs sampling is a special case of more general methods called Markov chain Monte Carlo (MCMC) methods
  - Metropolis-Hastings is one of the more famous MCMC methods (in fact, Gibbs sampling is a special case of Metropolis-Hastings)
- You may read about Monte Carlo methods – they're just sampling